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Prolongation Structure and Painlevé Property of the Gürses–Nutku Equations

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It is shown that the Gürses-Nutku equations have a finite prolongation algebra for any value of the parameter K. The Painlevé property of these equations is also examined.

A generalization of the KdV equation found by Gürses and Nutku (1981) is given by

$$u_t + 6uu_x + u_{xxx} - \lambda_x u_{xx} = 0 \tag{1a}$$

$$\lambda_t + 2u\lambda_x + 2Ku_x = 0 \tag{1b}$$

where K is an arbitrary constant. These equations arise as the embedding equations of a two-dimensional surface into a flat, three-dimensional space. Gürses and Nutku discussed the equivalence between the two-dimensional integrable systems and surface theory at the metric level. In a recent work, Chowdhury and Paul (1985) studied the prolongation structure of equations (1) without being aware of Gürses and Nutku (1981). They showed that the prolongation algebra closes for $K = -1, 1, 2$. In this work we restudy the prolongation algebra of Gürses-Nutku equations and show that it closes for any value of K. We also show that a nontrivial Bäcklund transformation cannot be found by prolongation techniques of Wahlquist and Estabrook (1975). We reduced the partial differential equations (1) to ordinary differential equations by transformation of variables and applied the Painlevé test of Ablowitz *et al.* (1980). We also applied the Painlevé test for partial differential equations introduced by Weiss *et al.* (1983) to the Gürses-Nutku equations. In both cases these equations fail when $K \neq 0$. Therefore they are not of P-type in their present form.

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Introducing the notation

$$u_x = p, \qquad p_x = q \tag{2}$$

we can write equations (1) as

$$u_t + 6up + q_x - \lambda_x q = 0 \tag{3a}$$

$$\lambda_t + 2u\lambda_x + 2Kp = 0 \tag{3b}$$

These first-order partial differential equations can be associated with the set of four 2-forms,

$$\begin{aligned}
\alpha_1 &= du \wedge dt - p\, dx \wedge dt \\
\alpha_2 &= dp \wedge dt - q\, dx \wedge dt \\
\alpha_3 &= dq \wedge dt + 6up\, dx \wedge dt - q\, d\lambda \wedge dt - du \wedge dx \\
\alpha_4 &= -d\lambda \wedge dx + 2u\, d\lambda \wedge dt + 2Kp\, dx \wedge dt
\end{aligned} \tag{4}$$

The set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ constitutes a closed ideal. That is,

$$d\alpha_a = \sum_{b=1}^{4} f_{ab}\alpha_b, \qquad a, b = 1, \ldots, 4 \tag{5}$$

where f_{ab} are some 1-forms. We now seek a set of 1-forms

$$W_k = dy_k + F_k\, dx + G_k\, dt, \qquad k = 1, \ldots, n \tag{6}$$

with $F_k(u, p, q, \lambda, y_k)$ and $G_k(u, p, q, \lambda, y_k)$ having the property that the prolonged ideal $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, w_1, \ldots, w_n\}$ is closed, that is,

$$dw_k = \sum_{a=1}^{4} g_{ka}\alpha_a + \sum_{i=1}^{n} n_k^i \wedge w^i \tag{7}$$

where n is the number of prolongation variables, and g_{ka} and n_k^i are some sets of 0-forms and 1-forms, respectively. This requirement gives the set of partial differential equations for F_k and G_k:

$$\begin{aligned}
F_{,p} = F_{,q} = 0, \qquad F_{,u} + G_{,q} &= 0 \\
G_{,\lambda} + 2uF_{,\lambda} - qF_{,u} &= 0 \\
pG_{,u} + qG_{,p} + 6upF_{,u} + 2KpF_{,\lambda} + [F, G] &= 0
\end{aligned} \tag{8}$$

where

$$[F, G] = \frac{\partial F}{\partial y^i} G^i - \frac{\partial G}{\partial y^i} F^i$$

The solutions of equations (8) are

$$\begin{aligned}
F &= u\beta_0 e^{-\lambda} + \gamma_0 e^{\lambda} \\
G &= -q\beta_0 e^{-\lambda} + p\delta_0 - 2u(u\beta_0 e^{-\lambda} + \gamma_0 e^{\lambda}) + \xi_0
\end{aligned} \tag{9}$$

where β_0, γ_0, δ_0, and ξ_0 are independent of u, p, q, and λ and satisfy the commutator relations

$$\begin{aligned}
&[\gamma_0, \beta_0] = \delta_0 \\
&[\delta_0, \gamma_0] = 2\gamma_0(K-1) \\
&[\delta_0, \beta_0] = 2\beta_0(1-K) \\
&[\xi_0, \gamma_0] = [\xi_0, \beta_0] = [\xi_0, \delta_0] = 0
\end{aligned} \tag{10}$$

The above algebra is finite-dimensional and no "closing off" is needed. With the choice $\xi_0 = 0$, which does not alter our conclusion, equations (9) and (10) reduce to the results given by Chowdhury and Paul (1985) for the special values $K = -1, 1, 2$, provided F and G in (9) are expressed in terms of

$$\rho = Ke^{\lambda/K} \tag{11}$$

Choosing $\gamma_0 = (1-K)X_1$, $\beta_0 = -X_2$, and $\delta_0 = (K-1)X_0$, one can write F and G in (9) in terms of the generators of $SL(2, R)$ algebra, satisfying the commutation relations

$$[X_0, X_1] = 2X_1, \qquad [X_0, X_2] = -2X_2, \qquad [X_1, X_2] = X_0 \tag{12}$$

Using the 2×2 matrix representation of these generators, we can construct an $SL(2, R)$-valued connection 1-form

$$\Gamma = \begin{pmatrix} \Theta_0 & \Theta_1 \\ \Theta_2 & \Theta_0 \end{pmatrix} \tag{13}$$

where

$$\begin{aligned}
&\Theta_0 = (K-1)u_x\, dt \\
&\Theta_1 = (1-K)e^{\lambda}(dx - 2u\, dt) \\
&\Theta_2 = -ue^{-\lambda}\, dx + e^{-\lambda}(2u^2 + u_{xx})\, dt
\end{aligned} \tag{14}$$

This connection defines a linear equation $d\Psi = -\Gamma\Psi$, where Ψ is a column vector with components Ψ_1 and Ψ_2. This associated linear equation does not contain any spectral parameter. To introduce such a parameter, we can perform an $SL(2, R)$ gauge transformation

$$\Gamma' = \Sigma\Gamma\Sigma^{-1} + \Sigma\, d\Sigma^{-1} \tag{15}$$

where $\det \Sigma = 1$. As a simple example, we choose

$$\Sigma = \begin{pmatrix} e^{i\xi x} & 0 \\ 0 & e^{-i\xi x} \end{pmatrix} \tag{16}$$

In the case the linear eigenvalue equations and the associated time evolution equations can be reduced to the following scattering problem:

$$\begin{aligned}\Psi_{xx} &= (2i\xi+\lambda_x)\Psi_x+[\xi^2-i\xi\lambda_x+(K-1)u]\Psi\\ \Psi_t &= [2ui\xi+(1-K)u_x]\Psi-2u\Psi_x\end{aligned} \tag{17}$$

Assuming that one particular solution of the prolonged ideal $\{\alpha_i, w_k\}$ is known, another solution of equations (1) can be written as

$$\begin{aligned}\tilde{u} &= \tilde{u}(u, p, q, y^i)\\ \tilde{\lambda} &= \tilde{\lambda}(\lambda, y^i)\\ \tilde{p} &= \tilde{p}(u, p, q, y^i)\\ \tilde{q} &= \tilde{q}(u, p, q, y^i)\end{aligned} \tag{18}$$

Substituting these into the set of forms

$$\begin{aligned}\tilde{\alpha}_1 &= d\tilde{u}\wedge dt-\tilde{p}\,dx\wedge dt\\ \tilde{\alpha}_2 &= d\tilde{p}\wedge dt-\tilde{q}\,dx\wedge dt\\ \tilde{\alpha}_3 &= d\tilde{p}\wedge dt+6\tilde{u}\tilde{p}\,dx\wedge dt-\tilde{q}\,d\tilde{\lambda}\wedge dt-d\tilde{u}\wedge dx\\ \tilde{\alpha}_4 &= -d\tilde{\lambda}\wedge dx+2\tilde{u}\,d\tilde{\lambda}\wedge dt+2K\tilde{p}\,dx\wedge dt\end{aligned} \tag{19}$$

and requiring these be in the ring of the prolonged ideal, we have the following set of differential equations:

$$\begin{aligned}\tilde{u}_{,q}=\tilde{p}_{,q}=\tilde{u}_{,p} &= o\\ p\tilde{u}_{,u}-\tilde{u}_{,y^i}F^i-\tilde{p} &= 0\\ p\tilde{p}_{,u}+q\tilde{p}_{,p}-\tilde{p}_{,y^i}F^i-\tilde{q} &= 0\\ p\tilde{q}_{,u}+q\tilde{q}_{,p}-6up\tilde{q}_{,q}+6\tilde{u}\tilde{p}-\tilde{q}_{,y^i}F^i+\tilde{q}\tilde{\lambda}_{,y^i}F^i-\tilde{u}_{,y^i}G^i &= 0\\ q\tilde{q}_{,q}-\tilde{q}\tilde{\lambda}_{,\lambda} &= 0\\ \tilde{q}_{,q}-\tilde{u}_{,u} &= 0\\ \tilde{\lambda}_{,\lambda}(\tilde{u}-u) &= 0\\ 2K(\tilde{p}-p\tilde{\lambda}_{,\lambda})-\tilde{\lambda}_{,y^i}G^i-2\tilde{u}\tilde{\lambda}_{,y^i}F^i &= 0\end{aligned} \tag{20}$$

The solutions of these equations are trivial, i.e.,

$$\tilde{u}=u, \qquad \tilde{\lambda}=\lambda, \qquad \tilde{q}=q, \qquad \tilde{p}=p \tag{21}$$

which means that there exists no Bäcklund transformation other than the identity map. This supports the claim that for nonlinear partial differential equations admitting only finite-dimensional prolongation algebra Bäcklund transformations other than the identity mapping seem to be absent (Leo *et al.*, 1983).

There is a close connection between solvable nonlinear evolution equations and nonlinear ordinary differential equations of Painlevé type. An ODE is said to possess the Painlevé property when all movable singularities are simple poles. Ablowitz *et al.* (1980) conjecture that a nonlinear ordinary differential equation obtained by an exact reduction of a nonlinear partial differential equation of inverse scattering transform class is either P-type or it must be related by a simple transformation to an ordinary differential equation that is P-type.

Introducing new variables

$$v = x^2 u, \qquad z = x^3/t \tag{22}$$

we can write equations (1) in the form of a single nonlinear ordinary differential equation,

$$\begin{aligned}&[27z^3 v''' + zv'(18v + 24 - z) - 12v(v+2)](6v - z)\\ &\quad - 6K(2v - 3v'z)(6v - 6v'z + 9z^2 v'') = 0\end{aligned} \tag{23}$$

where prime denotes derivative with respect to z. Using the algorithm for nonlinear ordinary differential equations introduced by Ablowitz *et al.* (1980), we obtain the following results:

Substituting

$$v \sim (z - z_0)^{-\alpha} v_0, \qquad z_0 \text{ is arbitrary} \tag{24}$$

into equation (23), we find

$$\alpha = -2, \qquad v_0 = -9z_0^2(K+2) \tag{25}$$

The resonances occur at $r_1 = -1$, $r_2 = 6$, $r_3 = 2(K+2)$.

It is obvious that for different values of K we have different resonance values. For $K \neq 0$, substituting

$$v = (z - z_0)^{-2} \sum_{j=0}^{6} v_j (z - z_0)^j \tag{26}$$

into the full equation (23) and requiring that the coefficients of $(z - z_0)^j$ must vanish identically, we find that v_j corresponding to the resonance values are not arbitrary. This means that at the resonances there must be logarithmic branch points. Therefore, equation (23) is not of P-type for $K \neq 0$.

A different exact reduction of equations (1) to an ODE may be obtained by looking for a self-similar solution

$$u = \frac{v(z)}{(3t)^{2/3}}, \qquad z = \frac{x}{(3t)^{1/3}} \tag{27}$$

where $v(z)$ satisfies

$$v'''(2v-z)+2Kv'v''+v'(12v^2-8vz+z^2)+2vz-4v^2=0 \tag{28}$$

This can be integrated once:

$$v''(2v-z)+(K-1)(v')^2+v'+4v^3-4v^2z+z^2v+C=0 \tag{29}$$

where C is an arbitrary integration constant.

This ODE is not of P-type, because the Painlevé expansion does not contain the correct number of arbitrary constants.

Another exact reduction of equations (1) to a different ODE may be obtained by looking for a traveling wave solution

$$u=v(z), \qquad z=x-ct \tag{30}$$

where $v(z)$ satisfies

$$v'''+6vv'-cv'+\frac{2Kv'}{2v-c}v''=0 \tag{31}$$

which can also be integrated once,

$$v''(2v-c)+(K-1)(v')^2+4v^3-4cv^2+c^2v-C'=0 \tag{32}$$

where C' is an arbitrary integration constant. Again this ODE does not have the correct number of arbitrary constant; hence, it is not of P-type for $K\neq 0$. This ODE admits solutions of the form of the elliptic functions, but the total number of arbitrary constants is reduced to two, instead of the three needed. The ODEs (23), (28), and (31) obtained by exact reductions of Gürses–Nutku equations, which are IST class, are not of P-type in their present form.

A different and more recent Painlevé test due to Weiss *et al.* (1983) is given as follows: A partial differential equation has the Painlevé property when its solutions are "single-valued" about the movable singularity manifold. If the singularity manifold is determined by

$$\Phi(x^0, x^1, \ldots, x^n)=0 \tag{33}$$

and $u^a(a=1,\ldots,N)$ satisfy a system of partial differential equations (N-equations), then the Painlevé expansion is given by

$$u^a=\Phi_a^\alpha \sum_{k=0}^{\infty} u^a_{(k)}(x^0, x^1, \ldots, x^n)\Phi^k \tag{34}$$

where $u^a_{(k)}$ are analytic functions of $(x^0, x^1, \ldots, x^n)$ in a neighborhood of the manifold (33). The substitution of (34) into the partial differential equations under consideration determines the possible values of α_a and gives the recursion relations for $u^a_{(k)}$.

A set of partial differential equations is said to have the Painlevé property in the sense of Weiss *et al.* provided the α_a are integers, the recursion relations are consistent, and the series expansion (34) contains the correct number of arbitrary functions. So there are basically three steps to the algorithm. Applying these steps to equations (1), we obtain the following:

1. Leading order analysis:

$$u \sim \Phi^{\alpha_1} u_0, \qquad \lambda \sim \Phi^{\alpha_2}\lambda_0,$$
$$\alpha_1 = -2, \qquad \alpha_2 = 2K \tag{35}$$
$$u_0 = -(2+K)\Phi_x^2, \qquad \lambda_0 = \Phi^{-2K}$$

Here K must take negative values in order for λ_0 to be analytic on $\Phi = 0$.

2. Resonances: Substituting

$$u = \Phi^{-2}u_0 + \beta_1\Phi^{r-2}, \qquad \lambda = \Phi^{2K}\lambda_0 + \beta_2\Phi^{r+2K} \tag{36}$$

into the leading terms of the original equation and requiring that β_1 and β_2 remain arbitrary, we have

$$(2+K)(r+2K)[(r-2)(r-3)(r-4) - 12(r-2) + 12(2+K)] = 0 \tag{37}$$

The roots of this equation determine the resonances. We must always have the root $r_1 = -1$, since it represents the arbitrariness of the singular manifold $\Phi = 0$. The other roots should be integers. $r_1 = -1$ is a root of the equation (37) when $K = 0$. For this case the other roots are found to be $r_2 = 0$, $r_3 = 4$, $r_4 = 6$. We know that the case $K = 0$ is related to the KdV equation, which is in fact P-type. When $K = 1/2$, $r_1 = -1$ is again a root of equation (37), but this time λ_0 is not analytic on $\Phi = 0$. Therefore equations (1) are not of P-type when $K \neq 0$.

In conclusion, we have showed that the coupled differential equations (1) have no nontrivial Bäcklund transformation. Even though the coupled partial differential equations (1) have a nontrivial prolongation structure and have a Lax pair, we have showed that they do not pass the Painlevé tests of Ablowitz *et al.* and Weiss *et al.* In that respect we should reexamine the Painlevé tests for coupled partial differential equations. Presumably we must expand each field, u and λ for our case, about distinct singular manifolds.

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